Nonlinear Dynamic Model of a Rotating Elastic Uniform Euler-Bernoulli Beam in 3D Space

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Abstract: A fully consistent geometrically nonlinear model for the elastic uniform Euler-Bernoulli beam with setting angle is proposed in this paper. Considering the Euler angles of the beam cross section in 3D space, the nonlinear partial differential equations of axial-bending-bending-torsional vibration of the cantilever beam are obtained by utilizing the principle of curvature and the Hamilton principle. The new mathematical model is more sophisticated than the previous models as a result of containing the centrifugal forces, Coriolis forces, rotary inertia, curvature items and other available terms. The complex mathematical model can be simplified easier by adjusting the setting angle and omitting some less important items. Comparisons are made in detail to existing models in the literature by analysis. More precise results may be obtained through the new models and formulations.

Keywords: Euler angles; rotating Euler-Bernoulli beam; axial-bending-bending-torsional coupling; gyroscopic forces

1. Introduction

Rotating cantilever beams are often used as a simple model for many engineering structures, such as propellers, compressor blades, turbine blades, and flexible satellite booms. For the high-speed rotating structures, they are often perturbed by unsteady aerodynamics. In addition, the temperature gradient and various loads can also lead to

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the deformation of the blades. Since the study of high speed rotating structures are very important, many scientists have studied the rotating structure modeling theoretically and experimentally.

Modeling issues regarding rotating beams have been interesting research topics for a long time. Carnegie [1] first derived the theoretical expression of potential and kinetic energy of rotating cantilever beam subjected to centrifugal loads. Others continued this research in-depth after that. Rao and Carnegie [2] considered the coupled bending-torsional vibration problem in the process of modeling while the axial vibration of the beam was not mentioned too much. Wright [3] considered the variation of mass distribution and analyzed its influence on vibration frequencies and modes of the beam. Researchers continued to take more factors into account, making the results more theoretically interesting. Bedoor [4] proposed a blade model with an elastic beam fixed on the hub by Lagrange method and finite element method considering the couple of the torsion and bending deformation. Yoo et al. [5] established the model of a rotating blade with lumped mass, and analyzed its vibration characteristics. Later they used the finite element method to solve the bending-bending-axial vibrations of the rotating beam neglecting the gyroscopic terms and torsion vibration.

Nassar and Bedoor [6] put forward the dynamic equation through Euler-Bernoulli beam theory as well as the small deformation theory, and studied the vibration of blade under the torsional deflection. The results obtained tend to be more accurate while Hamdan and Sinawib [7] simplified the blade to a slender flexible cantilever beam with a setting angle fixed on a rotating hub. Sakar and Sabuncu [8] studied the coupling effects of distance from the center of curvature to the center of mass, rotating speed, hub-radius and installation angle. Cai [9] has established a kind of coupled in-plane bending-stretching/compression vibration equation of Euler beam. The influence of rotational angular acceleration and the global moment of inertia is considered.

Diken and Alnefaie [10] studied the vibration of an unbalanced flexible rotor blade. The blade connected to the disc is considered to be a fixed free Euler Bernoulli beam. In their study, the coupled equations of the rotor blades are obtained by Lagrange equation. Some scholars put forward the nonlinear research and analysis of the blades. Farhadi and Hosseini [11] studied the characteristics of supersonic rotating rectangular plate, and establish the nonlinear equation by using the von Kármán first-order shear deformation theory considering the aerodynamic problems, plates' width ratio, thickness ratio, wheel radius ratio and rotation speed.

Hyungrae and Hong [12] presented a classic model of a rotating cantilever beam using nonlinear von Kármán strain and the corresponding linear stress while the rotary inertial terms were not considered. Ahmad and Mohammad [13] established the blade model based on the first order shear deformation theory (FSDT) and the classical beam theory (CLT), and the finite element method (FEM) is used to study the nonlinear viscoelastic model of the beam. Then, Yang and Zhang [14] focused on the research about nonlinear dynamics of axially moving beam with coupled longitudinal-transversal vibrations, and emphatically pointed out the gyroscopic coupling terms. Zhang and Feng [15] studied the nonlinear vibration of aero engine compressor blade, and the blade was simplified as a cantilever beam of thin functionally graded materials. The nonlinear partial differential equations of the blade were obtained by using the first-order piston theory, the effect of geometric large deformation, and Hamilton principle.

Based on the aforementioned references, it is found that the high-speed rotating blades are often simplified as beams or plates considering a series of factors in previous models. However, the models of Euler-Bernoulli beam in 3D space are still inadequate. In this work, we study systematically the dynamics of the rotating uniform Euler-Bernoulli beam with setting angle. The Euler angles, transformation of the beam cross section in 3D space and the principle of curvature are considered simultaneously for the first time. The nonlinear partial differential equations of axial-bending-bending-torsional vibration of the cantilever beam obtained in this paper based on the exact geometric relations are constructed and discussed. Comparisons in details are made with the existing models in the literature.

2. Formulation of the rotating Euler-Bernoulli beam in 3D space

2.1 Kinetic energy of the rotating Euler-Bernoulli beam

Consider a uniform Euler-Bernoulli beam fixed on a rigid hub of radius r as shown in Figure 1. To set up a inertial coordinate system $o\text{-}x^p\text{-}y^p\text{-}z^p$. Assuming the high speed

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rotating blade has a setting angle γ and rotating about its axis at a constant angular speed *Ω*. The undeformed beam geometry is described by the coordinate system *x*-*y*-*z*. The *x*-axis coincides with the centroid axis of the undeformed beam, and *y*-axis and *z*-axis are directions along the two sides of the undeformed beam cross-section. The relations between the non-inertial coordinate system *x*-*y*-*z* and the inertial coordinate system x^p-y^p-z^p are:

$$x = x^p, y = y^p \cos\gamma + z^p \sin\gamma, z = z^p \cos\gamma - y^p \sin\gamma \tag{1}$$

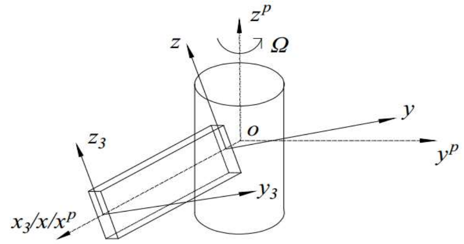

Fig. 1. Rotating beam configuration and coordinate system

Four directional vibrations of the beam will be studied including axial displacement *u* in the *x*-direction, two orthogonal lateral deflection: *v* in the *y*-direction, *w* in the *z*-direction, and the angle of twist *Φ*. We choose an arbitrary beam cross section to express the variety of Euler angles as shown in Figure 2.

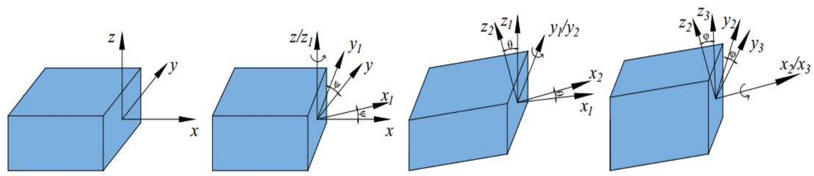

Fig. 2. Sequence of Euler angles.

From the undeformed plane, the cross section varies for three times:

1). Let the cross section rotate ψ about *z* axis from *x*-*y*-*z* to x_1-y_1-z_1:

$$\begin{pmatrix} i_{x_1} \\ i_{y_1} \\ i_{z_1} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_x \\ i_y \\ i_z \end{pmatrix}. \tag{2}$$

2). Let the cross section rotate θ about y_1 axis from x_1-y_1-z_1 to x_2-y_2-z_2:

$$\begin{pmatrix} i_{x_2} \\ i_{y_2} \\ i_{z_2} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} i_{x_1} \\ i_{y_1} \\ i_{z_1} \end{pmatrix}. \tag{3}$$

3). Let the cross section rotate φ about ε axis from x_2-y_2-z_2 to x_3-y_3-z_3:

$$\begin{pmatrix} i_{x_3} \\ i_{y_3} \\ i_{z_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i_{x_2} \\ i_{y_2} \\ i_{z_2} \end{pmatrix} = [Q] \begin{pmatrix} i_x \\ i_y \\ i_z \end{pmatrix}. \tag{4}$$

The transformation matrix **Q** can be defined as

$$[Q] = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \sin\varphi\cos\theta \\ \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi & \cos\varphi\sin\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\cos\theta \end{pmatrix}. \tag{5}$$

We use the Euler angles to relate the varieties of angular velocity through the above relations. The frame *x-y-z* which describe the undeformed beam is rotating at angular velocity Ω around the inertial frame while the relative angular velocity of the frame is $\dot{\psi}$ between x_1-y_1-z_1 and *x-y-z*, $\dot{\theta}$ between x_2-y_2-z_2 and x_1-y_1-z_1, $\dot{\varphi}$ between x_3-y_3-z_3 and x_2-y_2-z_2. The relative angular velocity to the inertial frame can be obtained as

$$\boldsymbol{\omega} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{pmatrix} \varphi - \psi\sin\theta + \Omega\sin\gamma\cos\theta\sin\psi - \Omega\cos\gamma\sin\theta \\ \theta\cos\varphi + \psi\sin\varphi\cos\theta + \Omega\sin\gamma\sin\varphi\sin\theta\sin\psi + \Omega\sin\gamma\cos\varphi\cos\psi + \Omega\cos\gamma\sin\varphi\cos\theta \\ -\dot{\theta}\sin\varphi + \dot{\psi}\cos\theta + \Omega\sin\gamma\cos\varphi\sin\theta\sin\psi - \Omega\sin\gamma\sin\varphi\cos\psi + \Omega\cos\gamma\cos\varphi\cos\theta \end{pmatrix}. \tag{6}$$

The values of cosine and sine are set unit and zero respectively by applying the small rotation angles assumption, and then we get

$$\boldsymbol{\omega} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{pmatrix} \dot{\varphi} + \Omega\sin\gamma v' + \Omega\cos\gamma w' \\ -\dot{w}' + \Omega\cos\gamma\varphi + \Omega\sin\gamma \\ \dot{v}' - \Omega\sin\gamma\varphi + \Omega\cos\gamma \end{pmatrix}, \tag{7}$$

$$Q = \begin{pmatrix} 1 & v' & w' \\ -v' & 1 & \varphi \\ -w' & -\varphi & 1 \end{pmatrix}. \tag{8}$$

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The directional time derivatives of $\mathbf{i}_{x_3}$, $\mathbf{i}_{y_3}$, $\mathbf{i}_{z_3}$ can be obtained as

$$\begin{Bmatrix} d\mathbf{i}_{x_3}/dt \\ d\mathbf{i}_{y_3}/dt \\ d\mathbf{i}_{z_3}/dt \end{Bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{i}_{x_3} \\ \mathbf{i}_{y_3} \\ \mathbf{i}_{z_3} \end{Bmatrix} = \begin{Bmatrix} \omega_3\mathbf{i}_{y_3} - \omega_2\mathbf{i}_{z_3} \\ -\omega_3\mathbf{i}_{x_3} + \omega_1\mathbf{i}_{z_3} \\ \omega_2\mathbf{i}_{x_3} - \omega_1\mathbf{i}_{y_3} \end{Bmatrix}. \tag{9}$$

Similarly, the directional time derivatives of $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ are

$$\begin{Bmatrix} d\mathbf{i}_x/dt \\ d\mathbf{i}_y/dt \\ d\mathbf{i}_z/dt \end{Bmatrix} = \begin{bmatrix} 0 & \Omega\cos\gamma & -\Omega\sin\gamma \\ -\Omega\cos\gamma & 0 & 0 \\ \Omega\sin\gamma & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{Bmatrix}. \tag{10}$$

These relations will be used in the following derivation of the kinetic energy.

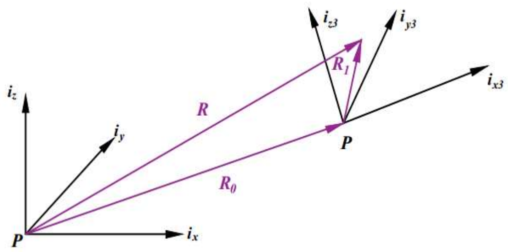

Fig. 3. Position vector of an arbitrary point

We express the deformation by choosing an arbitrary point on the beam denoted by the vector **R**, and it can be described as

$$\mathbf{R} = (R + x + u)\mathbf{i}_x + v\mathbf{i}_y + w\mathbf{i}_z + y\mathbf{i}_{y_3} + z\mathbf{i}_{z_3}. \tag{11}$$

Considering the time derivatives of the positions, the velocity of **R** is

$$\dot{\mathbf{R}} = \dot{u}\mathbf{i}_x + \dot{v}\mathbf{i}_y + \dot{w}\mathbf{i}_z + (R + x + u)\frac{d\mathbf{i}_x}{dt} + v\frac{d\mathbf{i}_y}{dt} + w\frac{d\mathbf{i}_z}{dt} + y\frac{d\mathbf{i}_{y_3}}{dt} + z\frac{d\mathbf{i}_{z_3}}{dt}. \tag{12}$$

We then express $\dot{\mathbf{R}}$ in the coordinate system of o-x-y-z as

$$\begin{aligned} \dot{\mathbf{R}} &= (\dot{u} - v\Omega\cos\gamma + w\Omega\sin\gamma)\mathbf{i}_x + \left[\dot{v} + (u + x + R)\Omega\cos\gamma\right]\mathbf{i}_y \\ &+ \left[\dot{w} - (u + x + R)\Omega\sin\gamma\right]\mathbf{i}_z + \begin{pmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\end{pmatrix}\mathbf{Q}^{\mathrm{T}}\mathbf{r}\boldsymbol{\omega} \end{aligned}, \tag{13}$$

where
$$\mathbf{r}=\begin{bmatrix} 0 & z & -y \\ -z & 0 & 0 \\ y & 0 & 0 \end{bmatrix}. \tag{14}$$

Then, we can obtain the kinetic energy of the rotating Euler-Bernoulli beam by substituting Eq. (13) into

$$T=\frac{\rho}{2}\int_0^L\int_A \dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\mathrm{d}A\mathrm{d}x\,, \tag{15}$$

where ρ is the density of the beam.

2.2 Potential energy of the rotating Euler-Bernoulli beam

Since $\mathbf{i}_{x_3}$, $\mathbf{i}_{y_3}$, $\mathbf{i}_{z_3}$ are unit vectors along the orthogonal curvilinear coordinate system x_3-y_3-z_3. We have

$$i_j\cdot i_k=\delta_{jk}\,. \tag{16}$$

Differentiating Eq. (16) with respect to x yields the identities

$$i_j'\cdot i_j=0,\quad i_j'\cdot i_k=-i_k'\cdot i_j;\quad j,k=1,2,3\,. \tag{17}$$

By using Eq. (17), we differentiate $\mathbf{i}_{x_3}$, $\mathbf{i}_{y_3}$, $\mathbf{i}_{z_3}$ with respect to x and obtain

$$\frac{\partial}{\partial x}\begin{pmatrix}\mathbf{i}_{x_3}\\ \mathbf{i}_{y_3}\\ \mathbf{i}_{z_3}\end{pmatrix}=\left[K\right]\begin{pmatrix}\mathbf{i}_{x_3}\\ \mathbf{i}_{y_3}\\ \mathbf{i}_{z_3}\end{pmatrix}. \tag{18}$$

where
$$\left[K\right]=\begin{bmatrix} i_{x_3}'\cdot i_{x_3} & i_{x_3}'\cdot i_{y_3} & i_{x_3}'\cdot i_{z_3} \\ i_{y_3}'\cdot i_{x_3} & i_{y_3}'\cdot i_{y_3} & i_{y_3}'\cdot i_{z_3} \\ i_{z_3}'\cdot i_{x_3} & i_{z_3}'\cdot i_{y_3} & i_{z_3}'\cdot i_{z_3} \end{bmatrix}=\begin{bmatrix} 0 & \rho_3 & -\rho_2 \\ -\rho_3 & 0 & \rho_1 \\ \rho_2 & -\rho_1 & 0 \end{bmatrix}, \tag{19}$$

and
$$\rho_1=i_{y_3}'\cdot i_{z_3},\quad \rho_2=i_{z_3}'\cdot i_{x_3},\quad \rho_3=i_{x_3}'\cdot i_{y_3}\,. \tag{20}$$

Here **[K]** is the curvature matrix, ρ_1 is the twisting curvature about the axis, ρ_2 is the bending curvature about y axis, and ρ_3 is the bending curvature about the z axis. Differentiating $\mathbf{i}_{y_3}$, $\mathbf{i}_{z_3}$ with respect to x, then we have

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$$\begin{aligned} i'_{y_3} &= Q'_{21} i_x + Q'_{22} i_y + Q'_{23} i_z \\ i'_{z_3} &= Q'_{31} i_x + Q'_{32} i_y + Q'_{33} i_z \end{aligned} \quad . \tag{21}$$

The differentiation is with respect to the undeformed length dx instead of the actual deformed length $(1+e)\, dx$. Substituting Q_{21}, Q_{22}, Q_{23} from Eq. (9) into Eq. (21). ρ_1, ρ_2, ρ_3 and e can be carried out by using Taylor expansion as

$$\begin{aligned} \rho_1 &= i'_{y_3} \cdot i_{z_3} = \sum_{i=1}^{3} Q'_{2i} Q_{3i} = \varphi' - \psi' \sin\theta = \varphi' + v'' w' \\ \rho_2 &= i'_{z_3} \cdot i_{x_3} = \sum_{i=1}^{3} Q'_{3i} Q_{1i} = \psi' \cos\theta \sin\varphi + \theta' \cos\varphi = -w'' + u'w'' + u''w' + v''\varphi \\ \rho_3 &= i'_{x_3} \cdot i_{y_3} = \sum_{i=1}^{3} Q'_{1i} Q_{2i} = \psi' \cos\theta \cos\varphi - \theta' \sin\varphi = v'' - u'v'' - u''v' + w''\varphi \\ e &= u' + \frac{1}{2}(1-u')\left(v'^2 + w'^2\right) + \cdots \end{aligned} \quad . \tag{22}$$

The potential energy for Euler-Bernoulli can be derived as follows.

$$U = \frac{1}{2}\left(\int_0^L EAe^2 dx + \int_0^L EI_y \rho_2{}^2 dx + \int_0^L EI_z \rho_3{}^2 dx + \int_0^L GI_p \rho_1{}^2 dx\right). \tag{23}$$

where E, G are the elastic and shear modulus, respectively, and I_y, I_z are the moment of inertia with respect to z and y axis. We finally substituting the resulted kinetic and potential energy into the Hamilton principle equation:

$$\int_{t_1}^{t_2} \left(\delta T - \delta U\right) \mathrm{d}t = 0\, . \tag{24}$$

2.3 Nonlinear dynamic equations of the rotating Euler-Bernoulli beam

We obtain the following four nonlinear governing equations.

axial directional displacement (x direction):

$$\begin{aligned} &\rho A\ddot{u} \underline{- 2\rho A\Omega \cos\gamma \dot{v} + 2\rho A\Omega \sin\gamma \dot{w}} - \rho A\Omega^2 u - \rho A\Omega^2 (\mathrm{R} + x) \\ &- EAu'' - EAv'v'' - EAw'w'' + EI_y(-w''w''' - w'w^{(4)} + 3u''w'w''' \\ &+ u'w''w''' + u'w'w^{(4)} + u^{(4)}w'^2 + 4u'''w'w'' + 2u''w''^2 + v^{(4)}w'\varphi + v'''w''\varphi \\ &+ v''w''\varphi' + 2v'''w'\varphi' + v''w'\varphi'') + EI_z(-v''v''' - v'v^{(4)} + 3u''v'v''' + u'v''v''' + u'v'v^{(4)} \\ &+ u^{(4)}v'^2 + 4u'''v'v'' + 2u''v''^2 - v'w^{(4)}\varphi - v''w'''\varphi - v''w''\varphi' - 2v'w'''\varphi' - v'w''\varphi'') = 0 \end{aligned} \quad . \tag{25}$$

chordwise directional displacement (y direction bending):

$$\begin{aligned}&\rho A\ddot{v}\underline{+2\rho A\Omega\cos\gamma\dot{u}}-\rho A\Omega^2\cos^2\gamma v\underline{+\rho A\Omega^2\cos\gamma\sin\gamma w}-\rho I_z\ddot{v}''\underline{+2\rho I_z\Omega\sin\gamma\dot{\varphi}'}\\&+\rho I_z\Omega^2 v''+\underline{\rho I_p\Omega^2\cos\gamma\sin\gamma w''}+2\rho I_y\Omega^2\sin^2\gamma v''-EA\left(u''v'+u'v''\right)\\&+EI_z(v^{(4)}-4u''v'''-2u'v^{(4)}-u^{(4)}v'-3u'''v''+w^{(4)}\varphi+2w'''\varphi'+w''\varphi'+2u''^2v''\\&+3u'u'''v''+4u'u''v'''+u'^2v^{(4)}+u''u'''v'+u'u^{(4)}v'-u''w'''\varphi-u'w''\varphi'-u'w^{(4)}\varphi\\&-2u'w'''\varphi'-u'w''\varphi'')+EI_y(-w^{(4)}\varphi-2w'''\varphi'-w''\varphi''+3u'''w''\varphi+3u''w'''\varphi+4u''w''\varphi'\\&+u'w^{(4)}\varphi+2u'w'''\varphi'+u'w''\varphi''+u^{(4)}w'\varphi+2u'''w'\varphi'+u''w'\varphi''+v^{(4)}\varphi^2+4v'''\varphi\varphi'+2v''\varphi'^2\\&+2v''\varphi\varphi'')+GI_p\left(v^{(4)}w'^2+4v'''w'w''+2v''w''^2+2v''w'w'''+\varphi'''w'+2\varphi''w''+\varphi'w'''\right)=0\end{aligned}. \tag{26}$$

flapwise directional displacement (z direction bending):

$$\begin{aligned}&\rho A\ddot{w}\underline{-2\rho A\Omega\sin\gamma\dot{u}}-\rho A\Omega^2\sin^2\gamma w\underline{+\rho A\Omega^2\cos\gamma\sin\gamma v}-\rho I_y\ddot{w}''\underline{+2\rho I_y\Omega\cos\gamma\dot{\varphi}'}\\&+\rho I_y\Omega^2 w''\underline{+\rho I_p\Omega^2\cos\gamma\sin\gamma v''}+2\rho I_z\Omega^2\cos^2\gamma w''-EA\left(u''w'+u'w''\right)\\&+EI_y(w^{(4)}-4u''w'''-2u'w^{(4)}-u^{(4)}w'-3u'''w''-v^{(4)}\varphi-2v'''\varphi'-v''\varphi'+2u''^2w''\\&+3u'u'''w''+4u'u''w'''+u'^2w^{(4)}+u''u'''w'+u'u^{(4)}w'+u''v'''\varphi+u'v''\varphi'+u'v^{(4)}\varphi\\&+2u'v'''\varphi'+u'v''\varphi'')+EI_z(v^{(4)}\varphi+2v'''\varphi'+v''\varphi''-3u'''v''\varphi-3u''v'''\varphi-4u''v''\varphi'\\&-u'v^{(4)}\varphi-2u'v'''\varphi'-u'v''\varphi''-u^{(4)}v'\varphi-2u'''v'\varphi'-u''v'\varphi''+w^{(4)}\varphi^2+4w'''\varphi\varphi'+2w''\varphi'^2\\&+2w''\varphi\varphi'')-GI_p\left(2v''v'''w'+v''^2w''+\varphi''v''+\varphi'v'''\right)=0\end{aligned}. \tag{27}$$

twist angle with respect to x axis:

$$\begin{aligned}&\rho I_p\ddot{\varphi}\underline{+2\rho I_z\Omega\sin\gamma\dot{v}'+2\rho I_y\Omega\cos\gamma\dot{w}'}-\rho I_z\Omega^2\sin^2\gamma\varphi-\rho I_y\Omega^2\cos^2\gamma\varphi\\&+\rho I_z\Omega^2\cos\gamma\sin\gamma-\rho I_y\Omega^2\cos\gamma\sin\gamma-GI_p\left(\varphi''+v'''w'+v''w''\right)\\&+EI_y\left(u'v''w''-v''w''+u''v''w'+2v''^2\varphi\right)+EI_z\left(-u'v''w''+v''w''-u''v'w''+2w''^2\varphi\right)=0\end{aligned}. \tag{28}$$

Now introducing dimensionless variables and parameters:

$$\begin{aligned}&\bar{x}=\frac{x}{L},\bar{u}=\frac{u}{L},\bar{v}=\frac{v}{L},\bar{w}=\frac{w}{L},\bar{R}=\frac{R}{L},\bar{t}=\frac{t}{T},T=\sqrt{\frac{\rho L^2}{E}},\\&\bar{\Omega}=T\Omega=\Omega\sqrt{\frac{\rho L^2}{E}},\eta_1=\frac{I_z}{AL^2},\eta_2=\frac{I_y}{AL^2},\eta=\frac{I_p}{AL^2},\mu=\frac{G}{E}\end{aligned}. \tag{29}$$

Then the four governing equations can be cast into the dimensionless form:

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$$
\begin{aligned}
&\ddot{\bar{u}}\underline{-2\bar{\Omega}\cos\gamma\dot{\bar{v}}+2\bar{\Omega}\sin\gamma\dot{\bar{w}}}-\bar{\Omega}^2\left(\bar{u}+\bar{R}+\bar{x}\right)-\bar{u}''-\bar{v}'\bar{v}''-\bar{w}'\bar{w}''+\eta_2(-\bar{w}''\bar{w}'''-\bar{w}'\bar{w}^{(4)}\\
&+3\bar{u}''\bar{w}'\bar{w}'''+\bar{u}'\bar{w}''\bar{w}'''+\bar{u}'\bar{w}'\bar{w}^{(4)}+\bar{u}^{(4)}\bar{w}'^2+4\bar{u}'''\bar{w}'\bar{w}''+2\bar{u}''\bar{w}''^2+\bar{v}^{(4)}\bar{w}'\varphi+\bar{v}'''\bar{w}''\varphi\\
&+\bar{v}''\bar{w}''\varphi'+2\bar{v}'''\bar{w}'\varphi'+\bar{v}''\bar{w}'\varphi'')+\eta_1(-\bar{v}''\bar{v}'''-\bar{v}'\bar{v}^{(4)}+3\bar{u}''\bar{v}'\bar{v}'''+\bar{u}'\bar{v}''\bar{v}'''+\bar{u}'\bar{v}'\bar{v}^{(4)}\\
&+\bar{u}^{(4)}\bar{v}'^2+4\bar{u}'''\bar{v}'\bar{v}''+2\bar{u}''\bar{v}''^2-\bar{v}'\bar{w}^{(4)}\varphi-\bar{v}''\bar{w}'''\varphi-\bar{v}''\bar{w}''\varphi'-2\bar{v}'\bar{w}'''\varphi'-\bar{v}'\bar{w}''\varphi'')=0
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{\bar{v}}\underline{+2\bar{\Omega}\cos\gamma\dot{\bar{u}}}-\bar{\Omega}^2\cos^2\gamma\bar{v}\underline{+\bar{\Omega}^2\cos\gamma\sin\gamma\bar{w}}-\eta_1\ddot{\bar{v}}''\underline{+2\eta_1\bar{\Omega}\sin\gamma\dot{\varphi}'}\\
&+\eta_1\bar{\Omega}^2\bar{v}''\underline{+\eta\bar{\Omega}^2\cos\gamma\sin\gamma\bar{w}''}+2\eta_2\bar{\Omega}^2\sin^2\gamma\bar{v}''-\bar{u}''\bar{v}'-\bar{u}'\bar{v}''+\eta_1(\bar{v}^{(4)}-4\bar{u}''\bar{v}'''\\
&-2\bar{u}'\bar{v}^{(4)}-\bar{u}^{(4)}\bar{v}'-3\bar{u}'''\bar{v}''+\bar{w}^{(4)}\varphi+2\bar{w}'''\varphi'+\bar{w}''\varphi'+2\bar{u}''^2\bar{v}''+3\bar{u}'\bar{u}'''\bar{v}''+4\bar{u}'\bar{u}''\bar{v}'''\\
&+\bar{u}'^2\bar{v}^{(4)}+\bar{u}''\bar{u}'''\bar{v}'+\bar{u}'\bar{u}^{(4)}\bar{v}'-\bar{u}''\bar{w}'''\varphi-\bar{u}''\bar{w}''\varphi'-\bar{u}'\bar{w}^{(4)}\varphi-2\bar{u}'\bar{w}'''\varphi'-\bar{u}'\bar{w}''\varphi'')\\
&+\eta_2(-\bar{w}^{(4)}\varphi-2\bar{w}'''\varphi'-\bar{w}''\varphi''+3\bar{u}'''\bar{w}''\varphi+3\bar{u}''\bar{w}'''\varphi+4\bar{u}''\bar{w}''\varphi'+\bar{u}'\bar{w}^{(4)}\varphi+2\bar{u}'\bar{w}'''\varphi'\\
&+\bar{u}'\bar{w}''\varphi''+\bar{u}^{(4)}\bar{w}'\varphi+2\bar{u}'''\bar{w}'\varphi'+\bar{u}''\bar{w}'\varphi''+\bar{v}^{(4)}\varphi^2+4\bar{v}'''\varphi\varphi'+2\bar{v}''\varphi'^2+2\bar{v}''\varphi\varphi'')\\
&+\mu\left(\bar{v}^{(4)}\bar{w}'^2+4\bar{v}'''\bar{w}'\bar{w}''+2\bar{v}''\bar{w}''^2+2\bar{v}''\bar{w}'\bar{w}'''+\varphi'''\bar{w}'+2\varphi''\bar{w}''+\varphi'\bar{w}'''\right)=0
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{\bar{w}}\underline{-2\bar{\Omega}\sin\gamma\dot{\bar{u}}}-\bar{\Omega}^2\sin^2\gamma\bar{w}\underline{+\bar{\Omega}^2\cos\gamma\sin\gamma\bar{v}}-\eta_2\ddot{\bar{w}}''\underline{+2\eta_2\bar{\Omega}\cos\gamma\dot{\varphi}'}\\
&+\eta_2\bar{\Omega}^2\bar{w}''\underline{+\eta\bar{\Omega}^2\cos\gamma\sin\gamma\bar{v}''}+2\eta_1\bar{\Omega}^2\cos^2\gamma\bar{w}''-\bar{u}''\bar{w}'-\bar{u}'\bar{w}''+\eta_2(\bar{w}^{(4)}-4\bar{u}''\bar{w}'''\\
&-2\bar{u}'\bar{w}^{(4)}-\bar{u}^{(4)}\bar{w}'-3\bar{u}'''\bar{w}''-\bar{v}^{(4)}\varphi-2\bar{v}'''\varphi'-\bar{v}''\varphi'+2\bar{u}''^2\bar{w}''+3\bar{u}'\bar{u}'''\bar{w}''+4\bar{u}'\bar{u}''\bar{w}'''\\
&+\bar{u}'^2\bar{w}^{(4)}+\bar{u}''\bar{u}'''\bar{w}'+\bar{u}'\bar{u}^{(4)}\bar{w}'+\bar{u}''\bar{v}'''\varphi+\bar{u}''\bar{v}''\varphi'+\bar{u}'\bar{v}^{(4)}\varphi+2\bar{u}'\bar{v}'''\varphi'+\bar{u}'\bar{v}''\varphi'')\\
&+\eta_1(\bar{v}^{(4)}\varphi+2\bar{v}'''\varphi'+\bar{v}''\varphi''-3\bar{u}'''\bar{v}''\varphi-3\bar{u}''\bar{v}'''\varphi-4\bar{u}''\bar{v}''\varphi'-\bar{u}'\bar{v}^{(4)}\varphi-2\bar{u}'\bar{v}'''\varphi'\\
&-\bar{u}'\bar{v}''\varphi''-\bar{u}^{(4)}\bar{v}'\varphi-2\bar{u}'''\bar{v}'\varphi'-\bar{u}''\bar{v}'\varphi''+\bar{w}^{(4)}\varphi^2+4\bar{w}'''\varphi\varphi'+2\bar{w}''\varphi'^2+2\bar{w}''\varphi\varphi'')\\
&-\mu\left(2\bar{v}''\bar{v}'''\bar{w}'+\bar{v}''^2\bar{w}''+\varphi''\bar{v}''+\varphi'\bar{v}'''\right)=0
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{\varphi}\underline{+\frac{2\eta_1}{\eta}\bar{\Omega}\sin\gamma\dot{\bar{v}}'+\frac{2\eta_2}{\eta}\bar{\Omega}\cos\gamma\dot{\bar{w}}'}-\frac{\eta_1}{\eta}\bar{\Omega}^2\sin^2\gamma\varphi-\frac{\eta_2}{\eta}\bar{\Omega}^2\cos^2\gamma\varphi\\
&+\frac{\eta_1}{\eta}\bar{\Omega}^2\cos\gamma\sin\gamma-\frac{\eta_2}{\eta}\bar{\Omega}^2\cos\gamma\sin\gamma-\mu\left(\varphi''+\bar{v}'''\bar{w}'+\bar{v}''\bar{w}''\right)\qquad\bullet\\
&+\eta_2\left(\bar{u}'\bar{v}''\bar{w}''-\bar{v}''\bar{w}''+\bar{u}''\bar{v}''\bar{w}'+2\bar{v}''^2\varphi\right)+\eta_1\left(-\bar{u}'\bar{v}''\bar{w}''+\bar{v}''\bar{w}''-\bar{u}''\bar{v}'\bar{w}''+2\bar{w}''^2\varphi\right)=0
\end{aligned}
\tag{30}
$$

By omitting the nonlinear terms, we can get the four simplified linear equations:

$$\ddot{\bar{u}}\underline{-2\bar{\Omega}\cos\gamma\dot{\bar{v}}+2\bar{\Omega}\sin\gamma\dot{\bar{w}}}-\bar{\Omega}^2\bar{u}-\bar{u}''=0$$

$$\begin{aligned}&\ddot{\bar{v}}\underline{+2\bar{\Omega}\cos\gamma\dot{\bar{u}}}-\bar{\Omega}^2\cos^2\gamma\bar{v}\underline{+\bar{\Omega}^2\cos\gamma\sin\gamma\bar{w}}-\eta_1\ddot{\bar{v}}''\underline{+2\eta_1\bar{\Omega}\sin\gamma\dot{\varphi}'}\\&+\eta_1\bar{\Omega}^2\bar{v}''\underline{+\eta\bar{\Omega}^2\cos\gamma\sin\gamma\bar{w}''}+2\eta_2\bar{\Omega}^2\sin^2\gamma\bar{v}''-\bar{u}_s''\bar{v}'-\bar{u}_s'\bar{v}''+\eta_1\bar{v}^{(4)}=0\end{aligned}$$

$$\begin{aligned}&\ddot{\bar{w}}\underline{-2\bar{\Omega}\sin\gamma\dot{\bar{u}}}-\bar{\Omega}^2\sin^2\gamma\bar{w}\underline{+\bar{\Omega}^2\cos\gamma\sin\gamma\bar{v}}-\eta_2\ddot{\bar{w}}''\underline{+2\eta_2\bar{\Omega}\cos\gamma\dot{\varphi}'}\\&+\eta_2\bar{\Omega}^2\bar{w}''\underline{+\eta\bar{\Omega}^2\cos\gamma\sin\gamma\bar{v}''}+2\eta_1\bar{\Omega}^2\cos^2\gamma\bar{w}''-\bar{u}_s''\bar{w}'-\bar{u}_s'\bar{w}''+\eta_2\bar{w}^{(4)}=0\end{aligned} \quad ,(31)$$

$$\begin{aligned}&\ddot{\varphi}\underline{+\frac{2\eta_1}{\eta}\bar{\Omega}\sin\gamma\dot{\bar{v}}'+\frac{2\eta_2}{\eta}\bar{\Omega}\cos\gamma\dot{\bar{w}}'}-\frac{\eta_1}{\eta}\bar{\Omega}^2\sin^2\gamma\varphi-\frac{\eta_2}{\eta}\bar{\Omega}^2\cos^2\gamma\varphi\\&+\frac{\eta_1}{\eta}\bar{\Omega}^2\cos\gamma\sin\gamma-\frac{\eta_2}{\eta}\bar{\Omega}^2\cos\gamma\sin\gamma-\mu\varphi''=0\end{aligned}$$

where

$$\bar{u}_s(x)=\bar{R}\cos\bar{\Omega}\bar{x}+\left(\frac{1}{\bar{\Omega}\cos\bar{\Omega}}+\bar{R}\tan\bar{\Omega}\right)\sin\bar{\Omega}\bar{x}-\bar{R}-\bar{x}\,. \quad (32)$$

3. Conclusions

The nonlinear partial differential equations of axial-bending-bending-torsional vibration in 3D space of the cantilever beam are obtained in this work. The underlined parts are gyroscopic terms due to the Coriolis force and Euler variation. The axial displacement and twisting displacement are gyroscopically coupled with both bending displacement motions. When the setting angle γ is set zero, the blade plane is parallel to the rotating plane. When the setting angle is 90°, the blade plane is in normal with the rotating plane. In the linear equations, when γ is zero, there will be gyroscopic coupling between the axial motion and chordwise transverse motion while not coupling with the flapwise motion, and the torsional vibrations couple with the flapwise motion only.

If the axial vibration and nonlinear terms are neglected, the similar governing equations by Huo [16] and Ozgumus [17] can be obtained. If we only consider the flapwise bending and axial vibrations, the governing equations of S.C. Lin [18-19] can be obtained. We can get the results of Invernizzi and Dozio [20] about the axial, edgewise, flapwise and torsional vibrations for the rotating Euler-Bernoulli beam by setting angle 0° or 90° and omitting nonlinear terms. By omitting the twist vibration, effects of curvature and

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the nonlinear terms, the full formulation of Huang [21] can also be obtained. We believe more precise results can be obtained through the new models and formulations. And we will study further in the future work.

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